A reduced-order model applied to the 2D shallow water equations

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Abstract

The numerical resolution of shallow water equations (SWE) is required in many environmental problems involving free surface flow. Reduced-order models (ROMs) based on the POD allow such problems to be solved more efficiently in terms of computational cost without losing accuracy, as discussed in this work.

2D SWE

The SWE system is formed by the depth averaged mass and momentum equations

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ q_x \\ q_y \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} q_x \\ \frac{q_x^2}{h}+\frac{gh^2}{2} \\ \frac{q_x q_y}{h} \end{pmatrix} + \frac{\partial}{\partial y}\begin{pmatrix} q_y \\ \frac{q_x q_y}{h} \\ \frac{q_y^2}{h}+\frac{gh^2}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ gh(S_{0x}-S_{fx}) \\ gh(S_{0y}-S_{fy}) \end{pmatrix},$$

where h is the water depth, $q_x = hu$ and $q_y = hv$ are the water discharges per unit width, and where u and v are velocities in the x- and y-direction; g is the gravitational acceleration; $S_{0x} = -\partial z/\partial x$ and $S_{0y} = -\partial z/\partial y$ are the bed slope being z the bed elevation; and $S_{fx} = n^2 u\sqrt{u^2+v^2}/h^{4/3}$ and $S_{fy} = n^2 v\sqrt{u^2+v^2}/h^{4/3}$ are the friction slope, following the Manning formula in terms of the roughness coefficient n.

Numerical model

The computational domain is discretized by means of $N_x \times N_y$ volume cells of uniform length Δx and Δy. The time step $\Delta t = t^{n+1} - t^n$ is selected using the Courant-Friedrichs-Lewy condition. The full-order model (FOM) is obtained using Godunov's scheme with Roe's numerical flux [3].

Reduced-order model

The POD-based ROM in this work is based on the snapshot method [4], which consists of the computation of a set of time numerical solutions of SWE, $\left(h_i^n, (q_x)_i^n, (q_y)_i^n\right)$, also called snapshots, as numerical approximations to h, q_x and q_y using the FOM. The snapshots are used to construct the POD basis of functions [1]. The POD-based intrusive ROM can be obtained from the 2D SWE by means of the Galerkin method [2]

$$h_{ij}^n \approx \sum_{k=1}^{N_{POD}} \hat{h}_k^n \phi_{ijk}, \quad \left(q_{x/y}\right)_{ij}^n \approx \sum_{k=1}^{N_{POD}} \left(\hat{q}_{x/y}\right)_k^n \left(\varphi_{x/y}\right)_{ijk},$$

where ϕ_{ijk} and $\left(\varphi_{x/y}\right)_{ijk}$ are the functions of the POD basis of h, q_x and q_y, respectively, being the number of POD modes $N_{POD} \ll N_x \times N_y$.

To increase the accuracy of the ROM solutions in non-linear problems, it is common to use time windows (proper interval decomposition, [5]), so that a POD basis is obtained in each time window.

In this work, the optimal values of the number of snapshots in each time window, N_s, and N_{POD} in terms of accuracy and CPU time are studied.

Numerical results

A circular dam-break in a domain with irregular bed

$$z(x,y,0) = \begin{cases} 0.2, & if\ r \le 1 \\ 0, & otherwise, \end{cases}$$

where $r^2 = (x-6)^2 + (y-6)^2$, is solved and the results computed by the POD-based ROM are compared with those of the FOM. The initial conditions (IC) are

$$h(x,y,0) = \begin{cases} 2, & if\ r \le 1 \\ 1, & otherwise, \end{cases} \quad u(x,y,0) = v(x,y,0) = 0.$$

Figure 1 shows the results of h, q_x and q_y at final time ($T = 0.818\ s$) obtained by the ROM, with $N_x = 200$, $N_y = 200$, $N_s = 2$ and $N_{POD} = 2$. The value of Manning's coefficient is $n = 0.033$.

Figure 2 on the left compares the ROM and FOM water depth solutions at $\boldsymbol{T}$; it also includes the IC and the bed level $\boldsymbol{z}$. As it can be seen, the ROM solution matches the FOM solution with very high accuracy. In addition, the required CPU time can be as much as two orders of magnitude less, as can be seen in Figure 2 on the right.

On the other hand, the variation of N_s and N_{POD} does not present substantial changes in ROM accuracy (measured with respect to the FOM by the L_2 norm), as it can be seen in Figure 2 on the center. Regarding CPU times, as shown in Figure 2 on the right, the higher N_s and the lower N_{POD}, the better efficiency levels are achieved.

Conclusions

The POD-based ROM applied in this work to 2D SWE with bed and friction source terms has proven to be as accurate as the FOM. Moreover, it is much more efficient in terms of CPU time, achieving improvements of two orders of magnitude.

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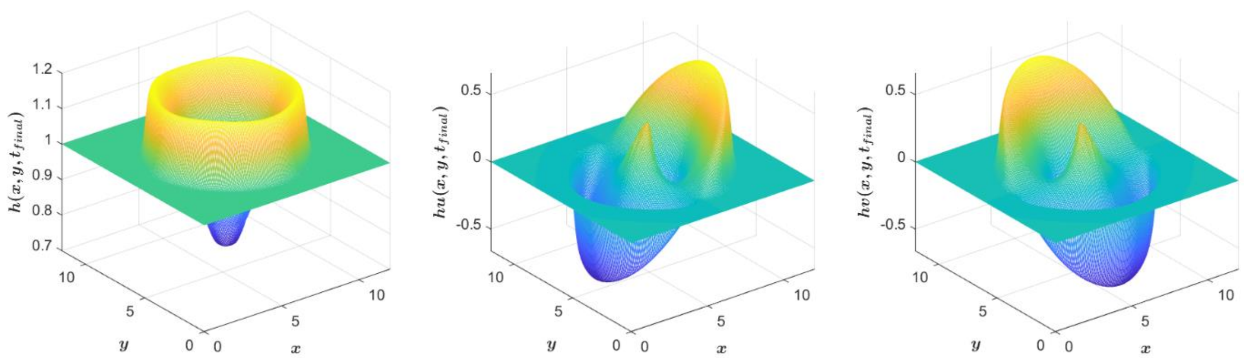

Figure 1. ROM solutions for h (left), q_x (center) and q_y (right) at T.

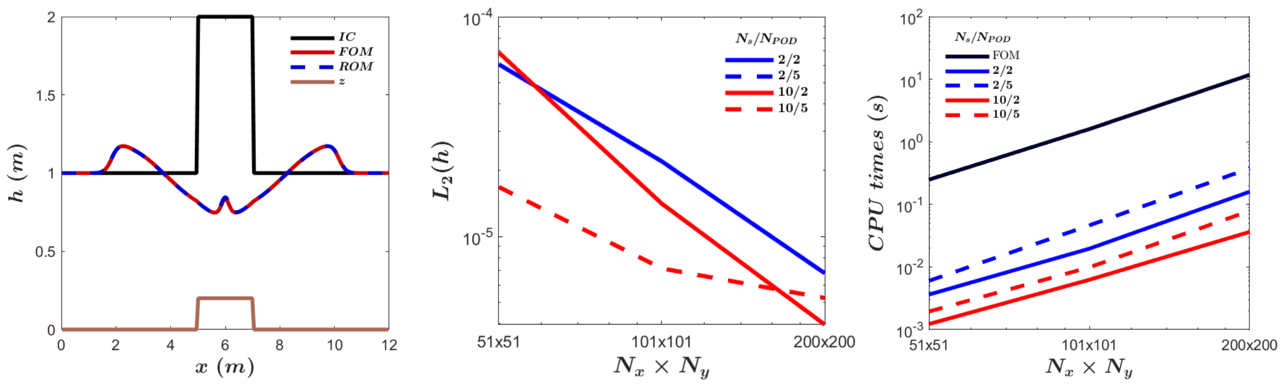

Figure 2. Comparison of FOM and ROM in terms of h solutions (left), accuracy (center) and CPU times (right).