A reduced-order model applied to the 2D unsteady shallow water equations

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Abstract

The numerical resolution of 2D unsteady shallow water equations (SWE) is required in many problems involving free surface flow. Reduced-order models (ROMs) based on the proper orthogonal decomposition (POD) allow such problems to be solved more efficiently in terms of computational cost without losing accuracy, as discussed in this work.

1. Introduction

The 2D SWE are widely used to model non-stationary free surface flows. The numerical resolution of such problems involves high computational costs in many cases. Among many others, POD-based ROMs [1] are one of the most popular choices to speed-up CPU times and maintain predictability.

The ROM strategy consists of two parts:

i) the off-line part, in which the ROM is trained following the snapshot method [4],

ii) and the on-line part, in which the ROM is solved up to the same training time at a reduced computational cost.

The sensitivity of the ROM to different parameters has been analysed to find their optimal values against the full-order model (FOM) in terms of computational efficiency and accuracy.

2. 2D SWE

The 2D SWE [3] is a non-linear system formed by the depth averaged mass and momentum equations

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ hu \\ hv \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} hu \\ hu^2 + gh^2/2 \\ huv \end{pmatrix} + \frac{\partial}{\partial y}\begin{pmatrix} hv \\ huv \\ hv^2 + gh^2/2 \end{pmatrix} = \begin{pmatrix} 0 \\ gh(S_{0x} - S_{fx}) \\ gh(S_{0y} - S_{fy}) \end{pmatrix} \qquad \text{(2D SWE)}$$

where

- h is the water depth;
- u and v are the water velocities;
- g is the gravitational acceleration;
- $S_{0x} = -\partial z_{bm}/\partial x$ and $S_{0y} = -\partial z_{bm}/\partial y$ are the bed slopes;
- z_{bm} is the bed elevation;
- $S_{fx} = n^2 u\sqrt{u^2+v^2}/h^{4/3}$ and $S_{fy} = n^2 v\sqrt{u^2+v^2}/h^{4/3}$ are the friction slopes
- and n is the Manning coeffcient.

3. Numerical model

The computational domain is discretized by means of $N_x \times N_y$ volume cells of uniform length Δx and Δy. The time step $\Delta t = t^{n+1} - t^n$ is selected using the Courant-Friedrichs-Lewy condition to ensure numerical stability. The FOM is obtained using an explicit Godunov's scheme with Roe's approximate Riemann solver flux [3].

The POD-based ROM in this work is based on the snapshot method [1, 4], a set of solutions at N_t different times of SWE by means of the FOM $\left\{\left\{h_{ij}^n\right\}, \left\{u_{ij}^n\right\}, \left\{v_{ij}^n\right\}\right\}$ such as, for example

$$\left(\begin{pmatrix} h_{11}^1 & h_{11}^2 & h_{11}^3 & h_{11}^4 & \cdots & h_{11}^{N_t-1} & h_{11}^{N_t} \\ h_{21}^1 & h_{21}^2 & h_{21}^3 & h_{21}^4 & \cdots & h_{21}^{N_t-1} & h_{21}^{N_t} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{N_x1}^1 & h_{N_x1}^2 & h_{N_x1}^3 & h_{N_x1}^4 & \cdots & h_{N_x1}^{N_t-1} & h_{N_x1}^{N_t} \end{pmatrix} \cdots \begin{pmatrix} h_{1N_y}^1 & h_{1N_y}^2 & h_{1N_y}^3 & h_{1N_y}^4 & \cdots & h_{1N_y}^{N_t-1} & h_{1N_y}^{N_t} \\ h_{2N_y}^1 & h_{2N_y}^2 & h_{2N_y}^3 & h_{2N_y}^4 & \cdots & h_{2N_y}^{N_t-1} & h_{2N_y}^{N_t} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{N_xN_y}^1 & h_{N_xN_y}^2 & h_{N_xN_y}^3 & h_{N_xN_y}^4 & \cdots & h_{N_xN_y}^{N_t-1} & h_{N_xN_y}^{N_t} \end{pmatrix}\right)$$

Each colour corresponds to a time window, following the PID method [5]. The snapshot method is applied on each time window individually, in this case consisting of two time instants.

The POD-based intrusive ROM can be obtained from the 2D SWE by means of the Galerkin method [2]

$$h_{ij}^n \approx \sum_{k=1}^{N_{POD}} \hat{h}_k^n \phi_{ijk},\ (hu)_{ij}^n \approx \sum_{k=1}^{N_{POD}} \hat{hu}_k^n \varphi_{ijk},\ (hv)_{ij}^n \approx \sum_{k=1}^{N_{POD}} \hat{hv}_k^n \Phi_{ijk}, \qquad \text{(Galerkin projection)}$$

where ϕ_{ijk}, φ_{ijk} and Φ_{ijk} are the functions of the POD bases of h, hu and hv, respectively, being the number of POD modes $N_{POD} \ll N_x \times N_y$.

In this work, the optimal values of the number of snapshots in each time window, N_s, and N_{POD} are sought in order to obtain accurate solutions in less CPU time than those of the FOM.

4. Numerical results

A circular dam-break with the following initial conditions (ICs)

$$h(x, y, 0\ s) = \begin{cases} 2\ m, & \text{if } r \le 1 \\ 1\ m, & \text{otherwise} \end{cases},\quad u(x, y, 0\ s) = v(x, y, 0\ s) = 0\ m/s,$$

is solved by means of the FOM and the POD-based ROM in a $12\ m \times 12\ m$ rectangular domain with irregular bed

$$z_{bm}(x, y) = \begin{cases} 0.2\ m, & \text{if } r \le 1 \\ 0\ m, & \text{otherwise} \end{cases},$$

where $r^2 = (x - 6\ m)^2 + (y - 6\ m)^2$. The value of Manning's coefficient is $n = 0.033\ s/m^{1/3}$ and the final computation time is $T = 0.818\ s$.

Figure 1 shows the results of water depth and water discharges at T obtained by the ROM, with $N_x = 200$, $N_y = 200$, two snapshots per time window, $N_s = 2$, and two POD modes solved, $N_{POD} = 2$.

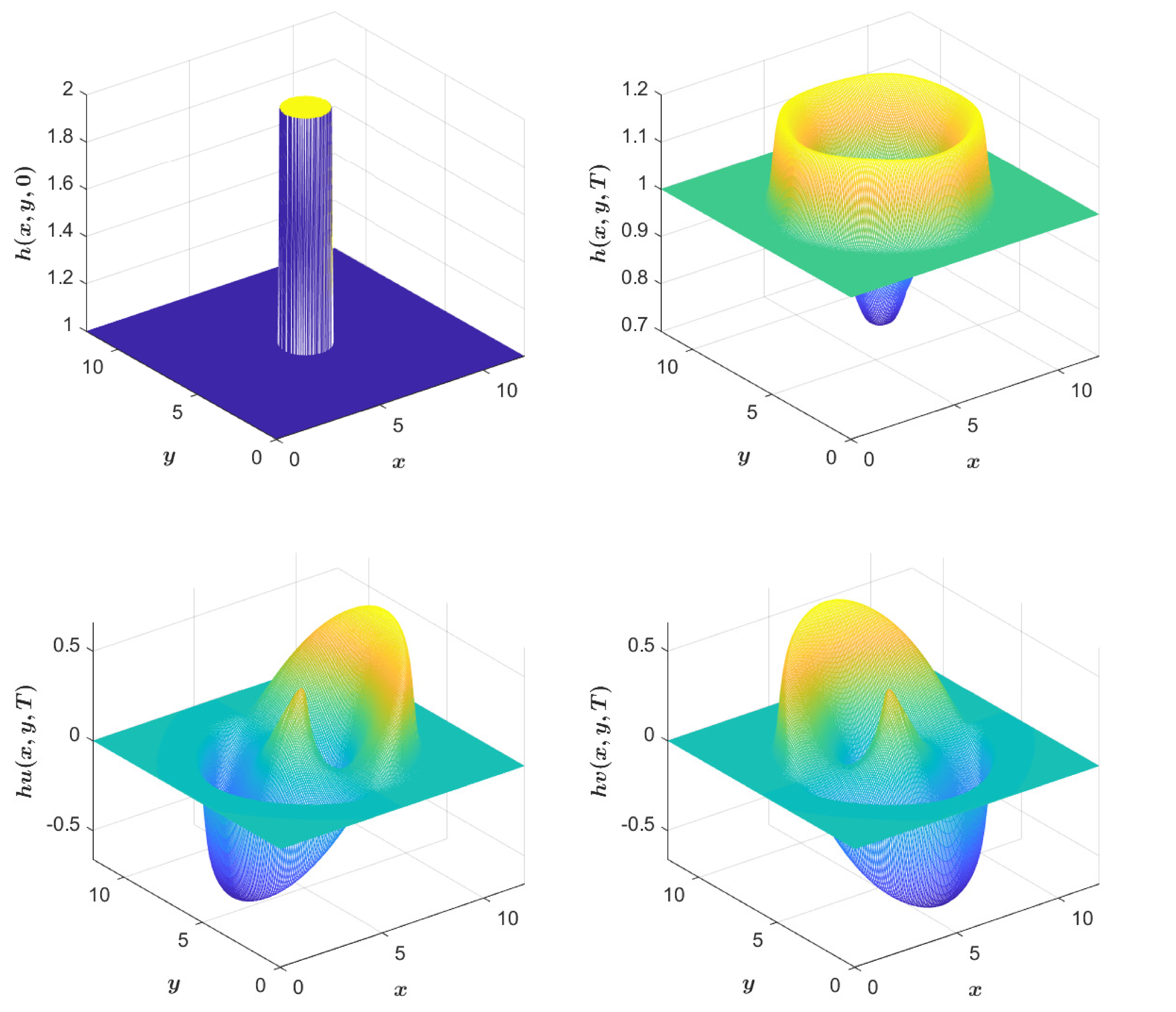

Figure 1: IC of water depth (top-left) and ROM solutions of water depth (top-right) and water discharges (bottom) at $T = 0.818\ s$.

Figure 2 on the left shows that the ROM solution matches the FOM solution with very high accuracy at T. In addition, the right plot in Figure 2 shows that the required CPU times by the ROM can be as much as two orders of magnitude less than that of the FOM.

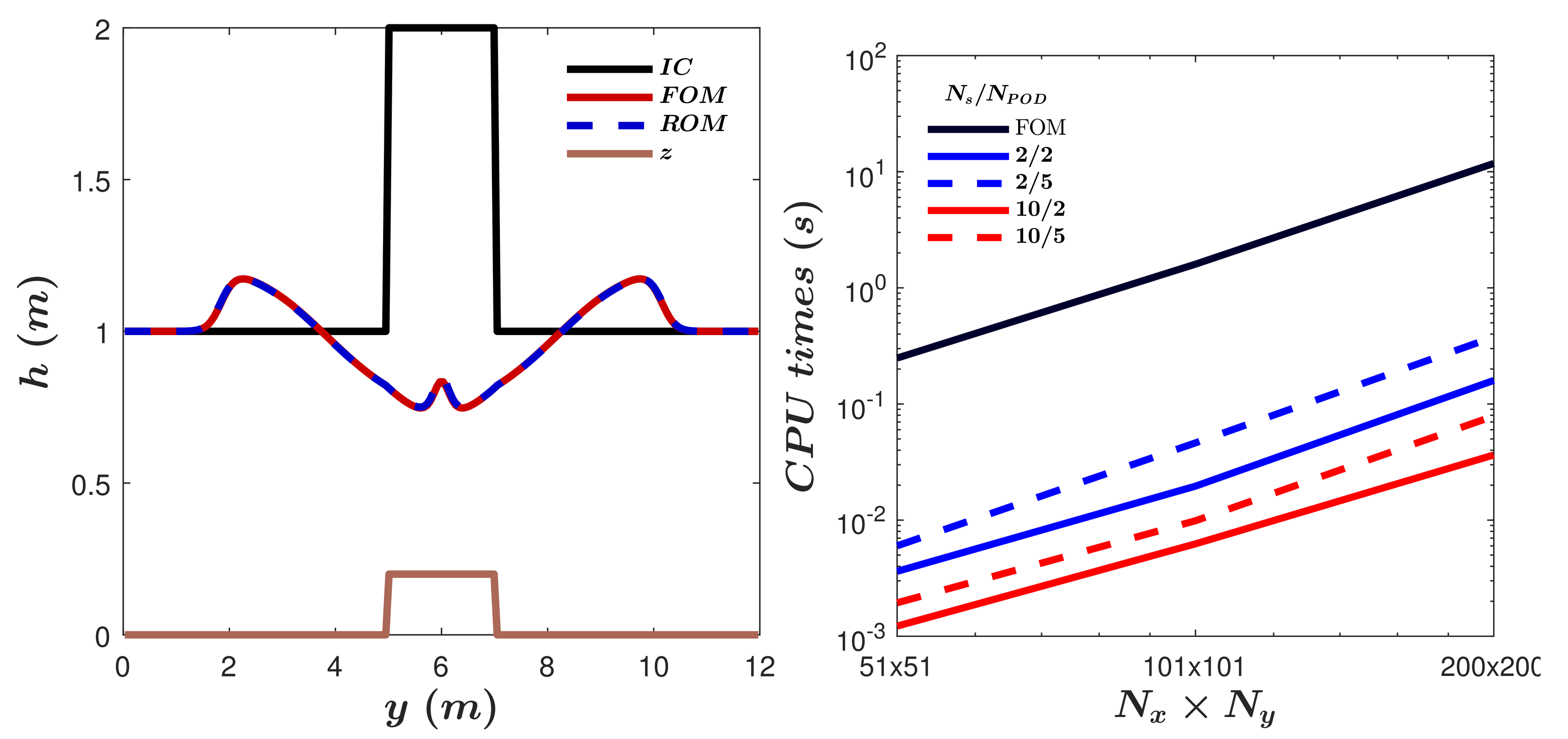

Figure 2: Comparison of FOM and ROM in terms of h solutions at T (left) and CPU times (right).

Conclusions

The POD-based ROM applied in this work to 2D SWE with bed and friction source terms has proven to be as accurate as the FOM. Moreover, it is much more efficient in terms of CPU time, achieving improvements of two orders of magnitude.

Further work

- Extend the FOM computation to unstructured meshes and solve properly wet/dry fronts.
- Use the POD method to calibrate parameters.

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