A New POD Method for Transport Equations

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Abstract

A novel implementation of the reduced-order model (ROM) of 1D advection-diffusion equation by means of a modified Proper Orthogonal Decomposition (POD) method is presented. This modified method is based on a coordinate transformation (CT-POD) that allows prediction beyond a training time horizon for advection dominated equations.

Introduction

The Proper Orthogonal Decomposition (POD) method is used to reduce the computational cost of numerical simulations by decomposing the random field of velocities of turbulent flows into a set of deterministic functions or modes (POD basis) [1, 2].

The POD reduced-order model (ROM) is designed to provide fast and accurate approximations of the full-order model (FOM) and can be obtained by means of the Galerkin method [3],

$$u(x,t) \approx \sum_{k=1}^{N_{POD}} \hat{u}_k(t)\phi_k(x),$$

where the maximum number of modes used is truncated to N_{POD}. In this work, we study the implementation of a modified POD to predict or extrapolate solutions in time.

POD method

Sirovich in 1987 [4] proposed the snapshots method in which solutions of the FOM (computed with a Finite volume method for instance) are collected at different times (snapshots matrix, S) to find the POD basis, $U = (\phi_1, \dots, \phi_{N_{POD}})$. This basis can be computed by means of the singular value decomposition:

$$SS^T = U\Sigma V^T$$

1D advection-diffusion equation

Let's consider the 1D advection-diffusion equation,

$$\frac{\partial u(x,t)}{\partial t} + a\frac{\partial u(x,t)}{\partial x} = \nu\frac{\partial^2 u(x,t)}{\partial x^2},$$

where $u(x,t)$ is the transported variable, a is the advective velocity and ν is the diffusion coefficient. The initial condition in the numerical experiments is a pulse centered at the point $x = d_0$.

The Péclet number is defined to be the ratio of the advection to the diffusion transport, $Pe = \frac{a\Delta x}{\nu}$, where Δx is the spatial mesh width of the FOM. Depending on the value of this number, the problem is advection or diffusion dominated.

Time extrapolation technique

The coordinate transformed POD (CT-POD) is based on the following coordinate transformation

$$\tilde{x} = \{\frac{d_0}{d(t)}\ if\ x \le d(t)\ \ L - \frac{L-d_0}{L-d(t)}(L-x)\ if\ x > d(t)$$

where L is the length of the domain, $u(x,t) = \tilde{u}(\tilde{x},t)$ and $d(t) = d_0 + at$ is the characteristic line with $0 < d_0 < L$.

Numerical method

We obtain the FOM by discretizing the 1D advection-diffusion equation; the ROM is obtained by introducing the Galerkin method into the FOM. The CT-FOM and CT-ROM are obtained by introducing the CT method into the FOM and ROM.

Numerical results

Figure 1 shows examples of POD (left) and CT-POD (right) solutions. The initial condition has been plotted together with the training time solution ($t = 0.1\ s$) and it is assumed that the pulse in the initial condition does not interact with the boundary of the domain. It is observed that the ROM cannot extrapolate in time, while the CT-ROM generates accurate approximations to the solutions for $t \geq 0.1\ s$.

Figure 2 shows the L_2 norm of the ROM with respect to the FOM (computed separately). In this results we can see that, in the case of high Péclet numbers, the ROM is not able to predict the solution at times greater than 0.1 s, whereas extrapolation in time is possible when the Péclet number is small. Besides that, the CT-ROM is able to predict solutions for both types of equations.

Conclusions

The CT-POD method allows the prediction of solutions in time with high accuracy for advection dominated equations, as well as higher computational efficiency.

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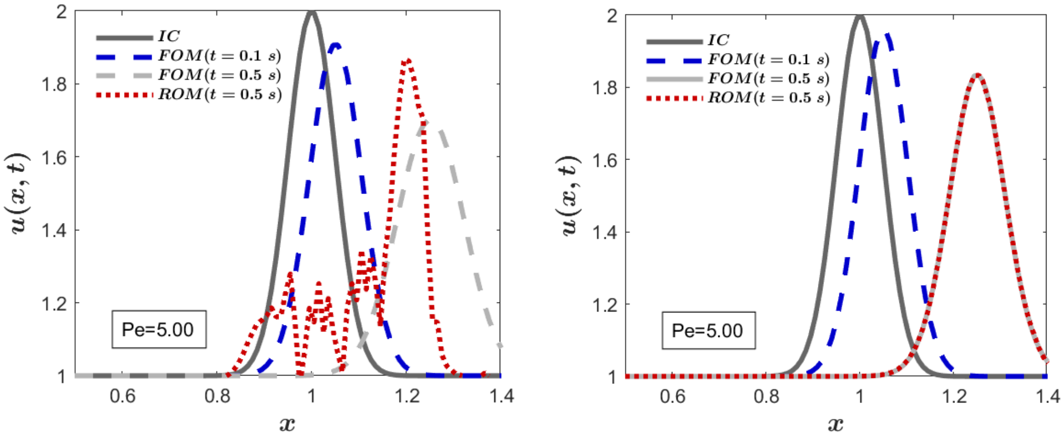

Figure 1. POD (left) and CT-POD (right) solutions extrapolation in time.

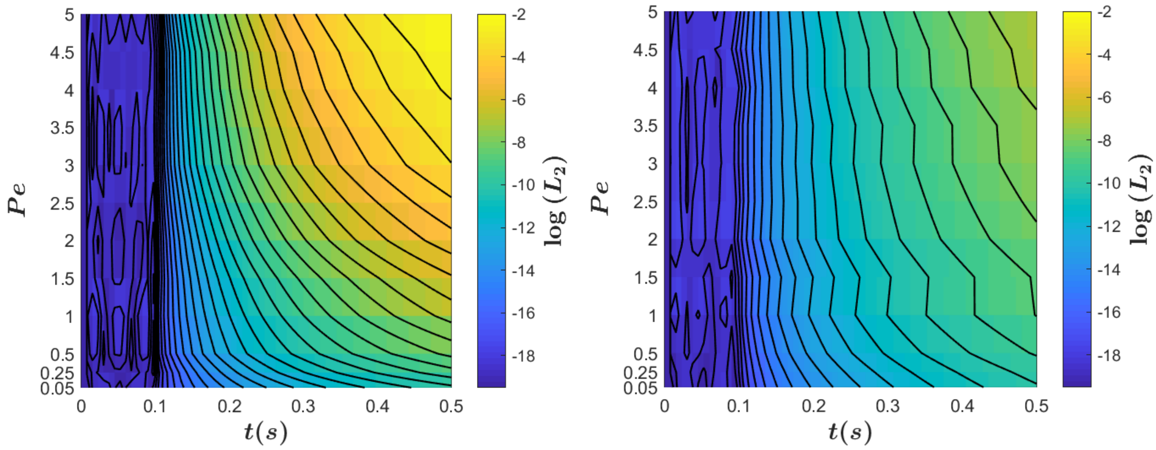

Figure 2. L_2 norms of POD (left) and CT-POD (right) solutions.

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