# POD-based ROM applied to unsteady free surface water flow


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## Abstract

The numerical resolution of shallow water equations (SWE) is required in many environmental problems involving free surface flows. The upwind augmented Roe's method [2] is widely used due to the robust and stable solutions it offers in realistic scenarios if properly corrected to deal with entropy fix and wet–dry fronts. On the other hand, reduced-order models (ROMs) are known to achieve more efficiency in terms of computational cost without losing accuracy. In this work, we analyse the properties and performance of a proper orthogonal decomposition (POD) based ROMs for this type of flows.


### 2D SWE

The 2D SWE system is a hyperbolic system of conservation laws formed by the depth averaged mass and momentum equations

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ q_x \\ q_y \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} q_x \\ \frac{q_x^2}{h}+\frac{gh^2}{2} \\ \frac{q_x q_y}{h} \end{pmatrix} + \frac{\partial}{\partial y}\begin{pmatrix} q_y \\ \frac{q_x q_y}{h} \\ \frac{q_y^2}{h}+\frac{gh^2}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ gh(S_{0x}-S_{fx}) \\ gh(S_{0y}-S_{fy}) \end{pmatrix},$$

where $h$ is the water depth, $q_x = hu$ and $q_y = hv$ are the water discharges per unit width, and where $u$ and $v$ are depth averaged velocities in the $x$- and $y$-direction; $g$ is the gravitational acceleration; $S_{0x} = -\partial z_b/\partial x$ and $S_{0y} = -\partial z_b/\partial y$ are the bed slope being $z_b$ the bed elevation; and $S_{fx} = n^2 u\sqrt{u^2+v^2}/h^{4/3}$ and $S_{fy} = n^2 v\sqrt{u^2+v^2}/h^{4/3}$ are the friction slopes, following the Manning formula in terms of the roughness coefficient $n$.

### Numerical model

The computational domain is discretized by means of $I_x \times I_y$ volume cells of uniform side length $\Delta x$ and $\Delta y$. The full-order model (FOM) is obtained using Godunov's explicit scheme with Roe's numerical flux. The time step $\Delta t = t^{n+1} - t^n$ is selected using the Courant-Friedrichs-Lewy condition. [2]

### Reduced-order model

The POD-based intrusive ROM can be obtained from the 2D SWE snapshots $\left(h_i^n, (q_x)_i^n, (q_y)_i^n\right)$ provided by the FOM using the Galerkin method

$$\boldsymbol{h}_{ij}^{n} \approx \sum_{k=1}^{M_{POD}} \hat{\boldsymbol{h}}_k^{n} \boldsymbol{\phi}_{ijk}, \quad \left(\boldsymbol{q}_{x/y}\right)_{ij}^{n} \approx \sum_{k=1}^{M_{POD}} \left(\hat{\boldsymbol{q}}_{x/y}\right)_k^{n} \left(\boldsymbol{\varphi}_{x/y}\right)_{ijk},$$

where $\phi_{ijk}$, $(\varphi_x)_{ijk}$ and $(\varphi_y)_{ijk}$ are the functions of the POD basis of $h$, $q_x$ and $q_y$, respectively, being the number of POD modes $M_{POD} \ll I_x \times I_y$.

### Numerical test case

The test case deals with the evolution of a dam-break wave over a dry bed with a triangular obstacle. The channel geometry is shown in Fig. 1 together with the initial condition (IC) of the water depth (initially at rest).

This case serves to test the performance of the ROM when the FOM includes the aforementioned numerical corrections to compute the training snapshots. The good properties are inherited by the ROM which does not include them.

The numerical results in Fig. 2 show that the ROM does not need to include entropy fix or wet–dry treatment to successfully reproduce realistic problems when it has been trained using solutions computed with a FOM in which these corrections are considered. The solutions of the ROM are very accurate when compared to those of the FOM, as can be seen in Fig.2. Furthermore, the speed-up achived by the ROM is 55 times.

## Conclusions

POD-based ROMs have proven to be very useful tools in solving transport problems such as the 2D SWE. The numerical test case helps to show that, if well designed, the ROM is able to reproduce realistic problems, since their solutions are accurate and are obtained in a more efficient way.

## REFERENCIAS


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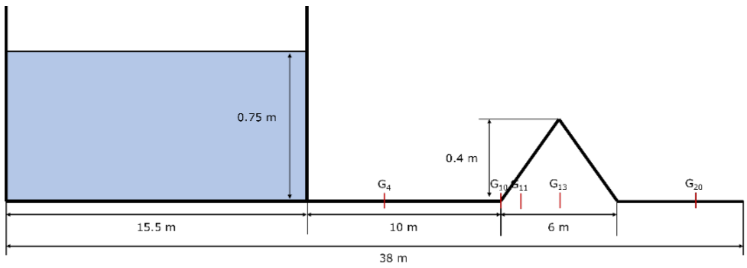


**Figure 1. Geometry of the channel.**

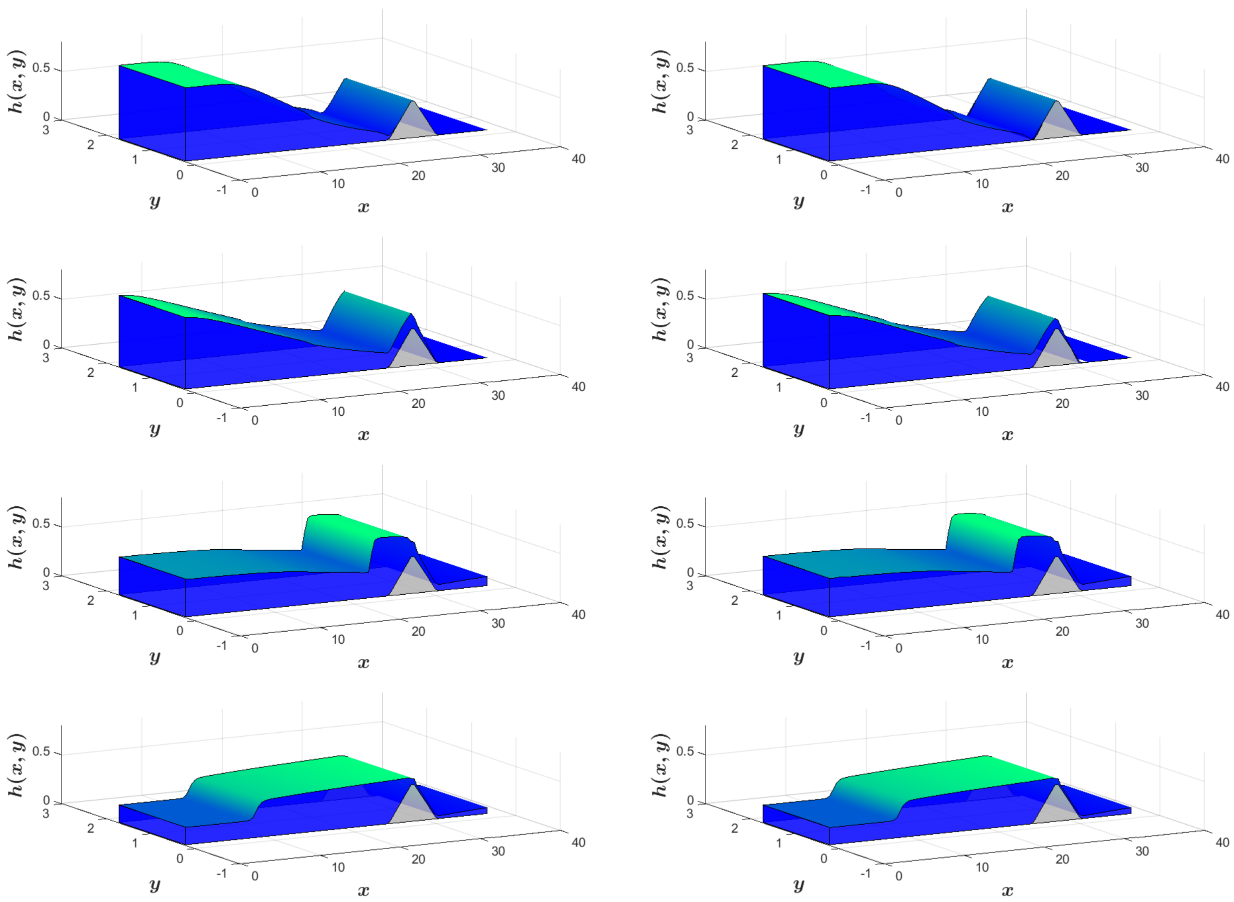

**Figure 2. FOM (left) and ROM (right) solutions for $h$ at different times (3 and 10 s) with 400 cells and 5 POD modes.**