POD-based ROM applied to unsteady free surface water flow

IV Jornada del I3A - XII Jornada de Jóvenes Investigadores e Investigadoras

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ABSTRACT The numerical resolution of shallow water equations (SWE) is required in many environmental problems involving free surface flows. The upwind augmented Roe's method [1] is widely used due to the robust and stable solutions it offers in realistic scenarios if properly corrected to deal with entropy fix and wet–dry fronts. On the other hand, reduced-order models (ROMs) are known to achieve more efficiency than the full-order models (FOMs) in terms of computational cost without losing accuracy. In this work, we analyse the properties and performance of a proper orthogonal decomposition (POD) based ROMs for this type of flows.

2D SWE

Water depth · Gravitational acceleration · Bed level · Water velocity: $u = u_x\hat{x} + u_y\hat{y}$

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ q_x \\ q_y \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} q_x \\ \frac{q_x^2}{h} + \frac{gh^2}{2} \\ \frac{q_x q_y}{h} \end{pmatrix} + \frac{\partial}{\partial y}\begin{pmatrix} q_y \\ \frac{q_x q_y}{h} \\ \frac{q_y^2}{h} + \frac{gh^2}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ -gh\left(\frac{\partial z_b}{\partial x} + \frac{n_b u_x \|\boldsymbol{u}\|}{h^{4/3}}\right) \\ -gh\left(\frac{\partial z_b}{\partial y} + \frac{n_b u_y \|\boldsymbol{u}\|}{h^{4/3}}\right) \end{pmatrix}$$

Water discharges in the x- and y-directions · Manning coefficient

Godunov method with Roe numerical fluxes.

FOM

Time step satisfying the CFL condition · Cell edges contributions. For more info, see [2]. · Number of time steps

$$\boldsymbol{U}_{ij}^{n+1} = \boldsymbol{U}_{ij}^{n} - \frac{\Delta t}{\Delta x \Delta y}\sum_{w_{ij}}^{4}\sum_{m}^{3}\left[\tilde{\lambda}^{-}\left(\tilde{\alpha} - \frac{\tilde{\beta}}{\tilde{\lambda}}\right)\boldsymbol{e}\right]_{w_{ij}}^{m,n}, \qquad \begin{matrix} n = 1, \dots, N_t \\ i = 1, \dots, I_x \\ j = 1, \dots, I_y \end{matrix}$$

$$\boldsymbol{U}_{ij}^{n} = \begin{pmatrix} h_{ij}^{n} \\ (q_x)_{ij}^{n} \\ (q_y)_{ij}^{n} \end{pmatrix}$$

Cell sizes in the x- and y-directions · Number of volume cells in the x- and y-directions

Galerkin method: $\boldsymbol{U}_{ij}^{n} \approx \sum_{k=1}^{M_{POD}} \widehat{\boldsymbol{U}}_k^n \boldsymbol{\phi}_{ijk}$,

where $\boldsymbol{\phi}_{ijk} = \left((\phi_h)_{ijk}, \ (\phi_{q_x})_{ijk}, \ (\phi_{q_y})_{ijk}\right)^T$ are the functions of the POD basis.

ROM

$$\widehat{\boldsymbol{U}}_k^{n+1} = \widehat{\boldsymbol{U}}_k^{n} + \frac{\Delta t}{\Delta x \Delta y} M \widehat{\boldsymbol{U}}_k^{n} + \Delta t N, \qquad k = 1, \dots, M_{POD}$$

$$\widehat{\boldsymbol{U}}_k^{n} = \begin{pmatrix} \hat{h}_k^{n} \\ (\hat{q}_x)_k^{n} \\ (\hat{q}_y)_k^{n} \end{pmatrix}$$

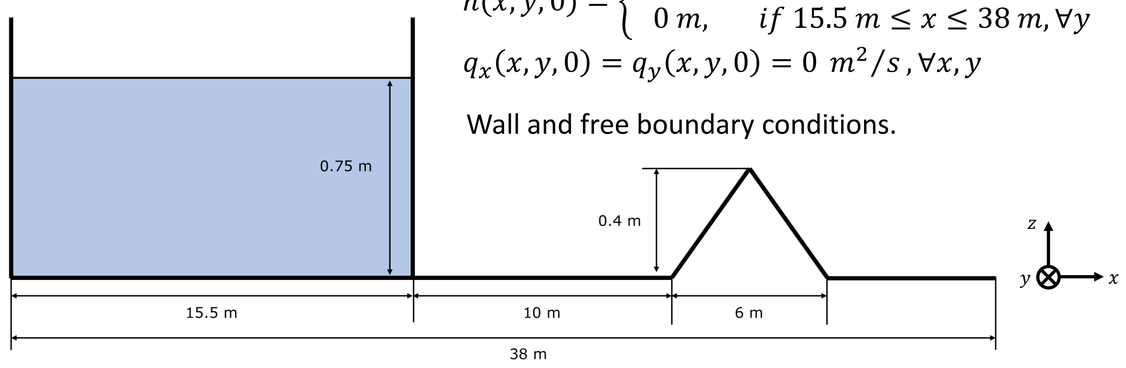
Matrices precomputed by applying the POD to the training snapshots. For more info, see [3]. · Number of POD modes.

Test case: geometry and initial conditions

$$h(x,y,0) = \begin{cases} 0.75\ m, & if\ 0\ m \le x < 15.5\ m, \forall y \\ 0\ m, & if\ 15.5\ m \le x \le 38\ m, \forall y \end{cases}$$

$$q_x(x,y,0) = q_y(x,y,0) = 0\ m^2/s\,, \forall x, y$$

Wall and free boundary conditions.



Numerical corrections to ensure a robust FOM:
- Entropy fix
- Wet-dry treatment
- Positive water depths

For more info, see [2].

FOM solution: training snapshots

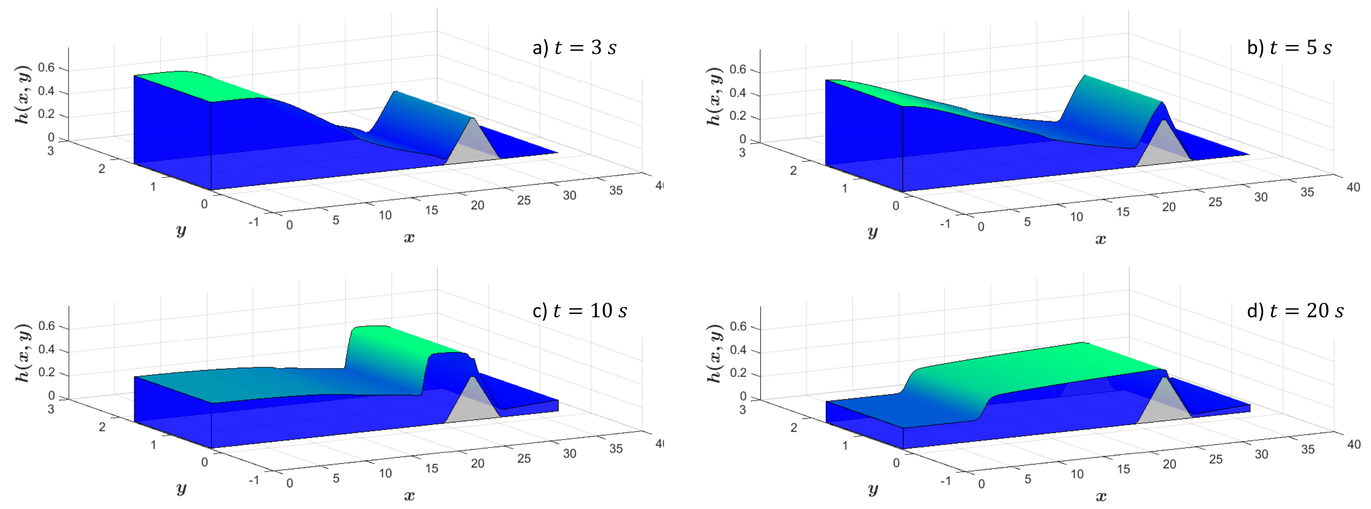


Figure 2. FOM solution of h at different times with $I_x = 100$, $I_y = 4$.

Inverse Galerkin method

Final ROM solution

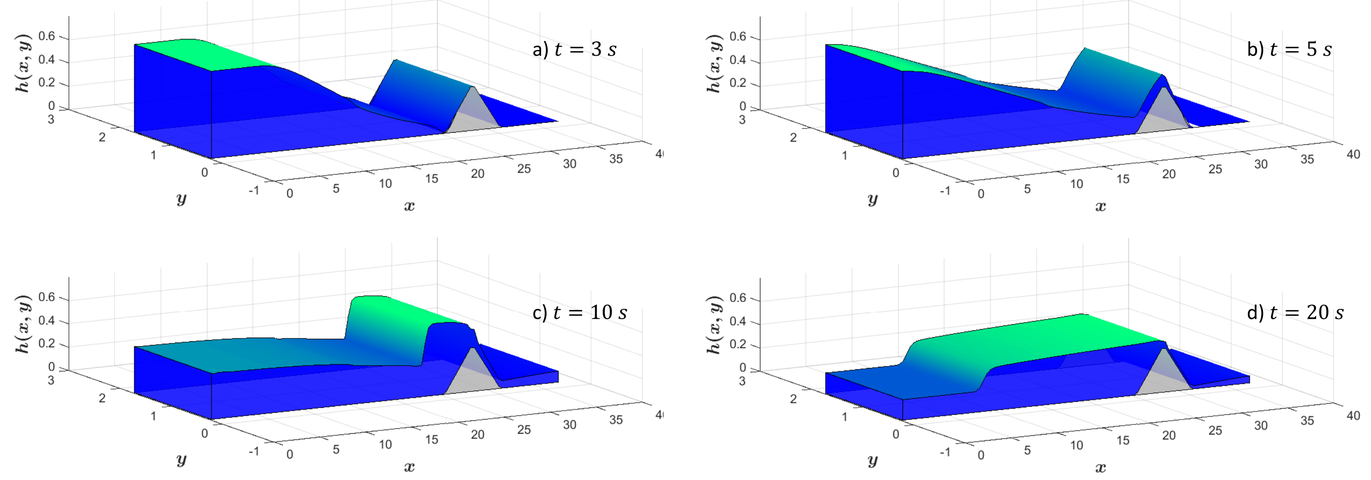


Figure 3. ROM solution of h at different times with $M_{POD} = 5$.

CONCLUSIONS The ROM can accurately (**Fig. 2** & **3**) and more efficiently (**Tab.**) reproduce realistic problems than the FOM, including the numerical corrections only in the training snapshots, but not in the resolution.

$I_x \times I_y$	100×4	200×4	400×4	600×4
τ_{CPU}^{FOM} (s)	1.87	6.61	25.71	59.06
τ_{CPU}^{ROM} (s)	$3.06 \cdot 10^{-2}$	$3.06 \cdot 10^{-2}$	$3.06 \cdot 10^{-2}$	$3.06 \cdot 10^{-2}$
Speed-up	**60.98**	**54.89**	**54.14**	**54.30**

Table. CPU times required by different mesh refinements and the corresponding speed-ups, defined as $\tau_{CPU}^{FOM}/\tau_{CPU}^{ROM}$.

References

[1] P. García-Navarro and M.E. Vázquez-Cendón. On numerical treatment of the source terms in the shallow water equations. Comput. Fluids 2000, 29(8), 951–979.

[2] I. Echeverribar, M. Morales-Hernández, P. Brufau, P. García-Navarro, 2D numerical simulation of unsteady flows for large scale floods prediction in real time, Adv. Water Resour. 134 (2019) 103444.

[3] P. Solán-Fustero, J.L. Gracia, A. Navas-Montilla and P. García-Navarro. Development of POD-based Reduced Order Models applied to shallow water equations using augmented Riemann solvers. Comput. Methods Appl. Mech. Engrg., 410 (2023) 116038.