Depth-Averaged Rheological Models for Landslide and Complex Flow Simulations on Steep Slopes with Non-Hydrostatic Pressure

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Introduction & Motivation

Depth-averaged models, such as the **Shallow Water Equations (SWE)**, are widely used for geophysical flows (**lahars**, **debris flows**, **avalanches**) due to their low computational cost compared to 3D models. However, standard simplifications in friction laws can cause simulation results to diverge during complex events. In this work, we revisit two key aspects affecting the derivation of these closure relations: the **global-local coordinate system** and **non-hydrostatic effects**.

Local coordinates *vs* Global coordinates

For steep slopes, like mountainous regions, shallow water equations may produce wrong results for $\theta > 10º$ [1]

Local

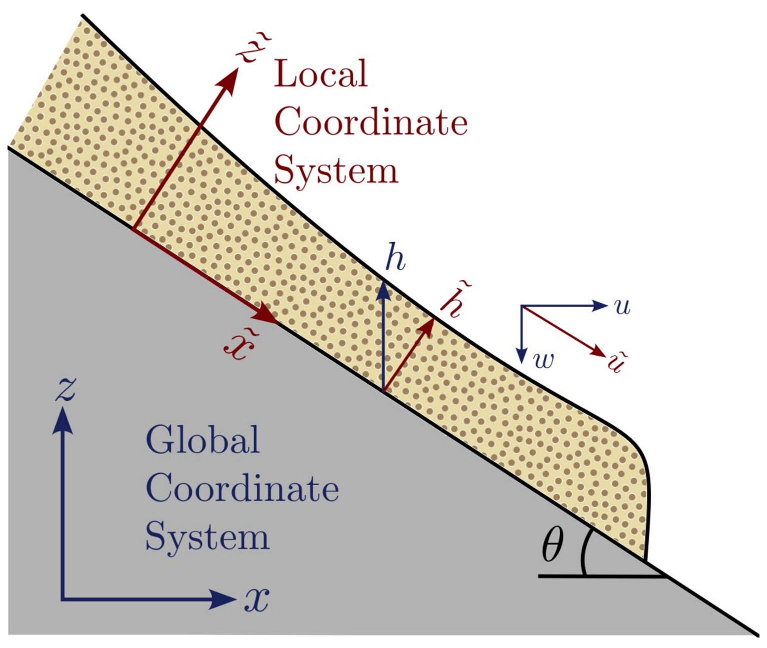
$$\rho\left(\frac{\partial\tilde{u}}{\partial t}+\tilde{u}\frac{\partial\tilde{u}}{\partial\tilde{x}}+\tilde{w}\frac{\partial\tilde{u}}{\partial\tilde{z}}\right)=-\frac{\partial p}{\partial\tilde{x}}+\frac{\partial\tau_{\tilde{x}\tilde{z}}}{\partial\tilde{z}}+\rho g\sin\theta$$

Global

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\frac{\partial\tau_{xz}}{\partial z}$$

Pressure

$$\frac{\partial p}{\partial z}=-\rho g-\underbrace{\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}\right)}_{p_{nh}(z)}$$

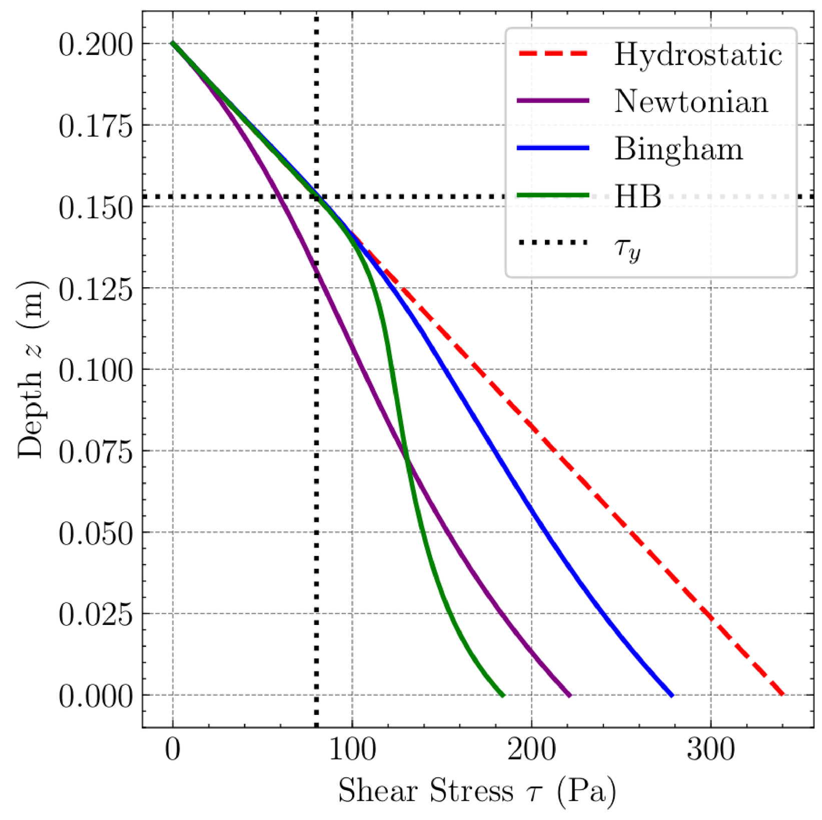


From Navier-Stokes to the Shear profile

$\tau(z)$ is derived from the X momentum equation [2]:

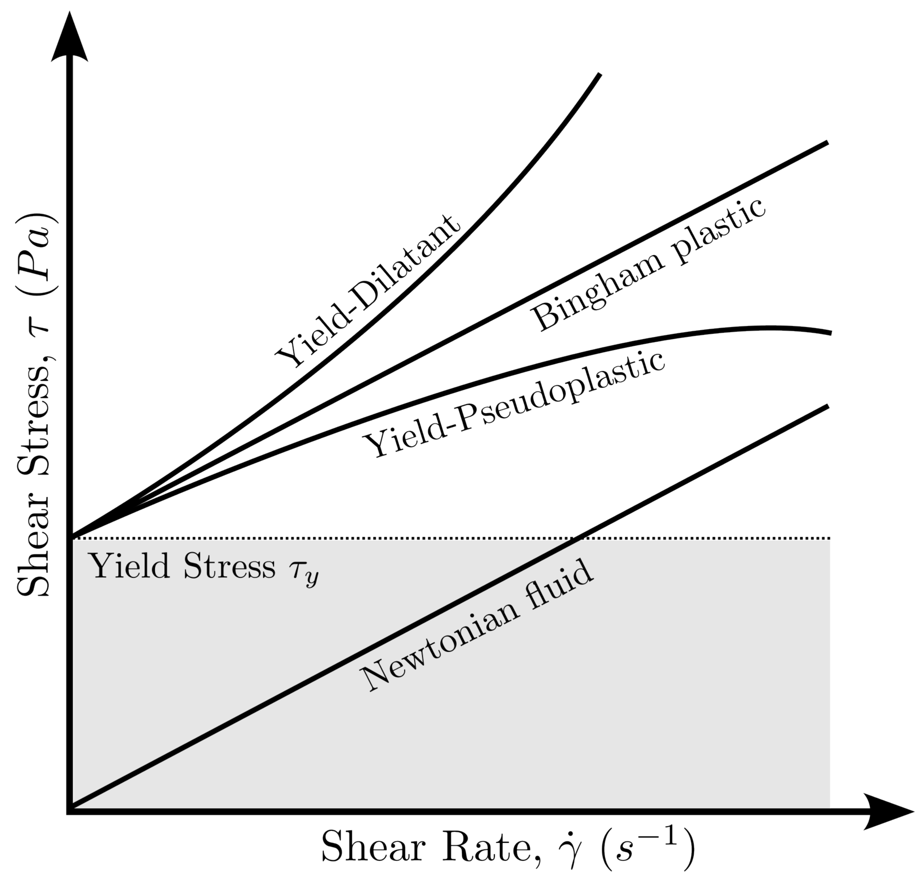
- **Steady** and **Uniform** flow assumptions: $\frac{\partial u}{\partial t}=0, \frac{\partial u}{\partial x}=0$
- Simplified shear stress: $\tau_{xz}\gg\tau_{xx}$
- **Hydrostatic** pressure: $p(z)=\rho g(h_s-z), w\approx 0$

$$\frac{\partial\tau}{\partial z}=-\rho g\frac{\partial h_s}{\partial x}=-\rho g\ \tan\theta$$
$$\Downarrow$$
$$\tau(z)=\rho g\tan\theta\,(h_s-z)$$



Constitutive models for friction

- Newtonian: $\tau=\mu\dot\gamma=\mu\frac{du}{dz}$
- Bingham plastic: $\tau=\tau_y+\mu_b\frac{du}{dz}$
- Herschel-Bulkley: $\tau=\tau_y+K\left(\frac{du}{dz}\right)^n$



Shear Profile Integration $\tau(z)$

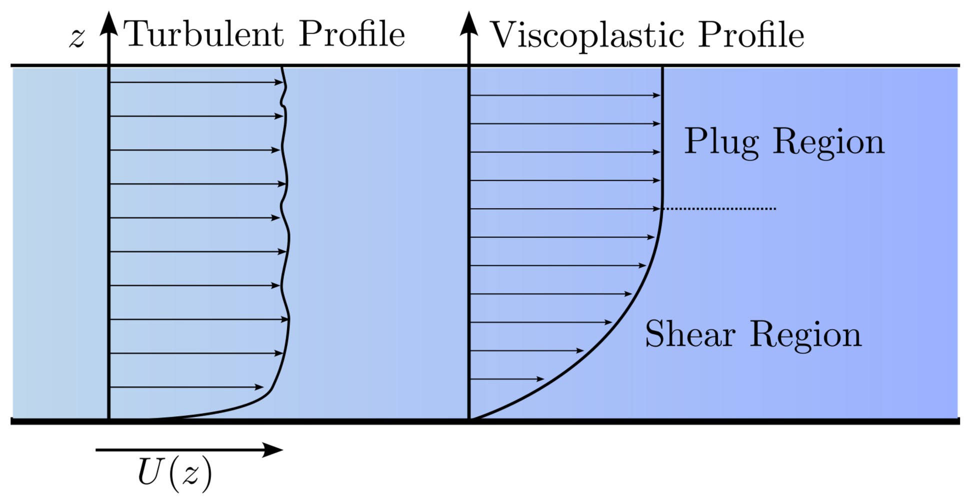
$$\frac{du}{dz}=\left(\frac{\tau-\tau_y}{K}\right)^{1/n}$$
$$\Downarrow$$
$$\begin{cases}\dfrac{hn}{K^{\frac{1}{n}}\tau_b(1+n)}\left[(\tau-\tau_y)^{\frac{n+1}{n}}-\left(\tau_b-\tau_y-\tau_b\dfrac{z}{h}\right)^{\frac{n+1}{n}}\right] & if\ z<z_p\\[2ex] \dfrac{hn\tau_b^{1/n}}{K^{\frac{1}{n}}(1+n)} & if\ z>z_p\end{cases}$$

Velocity profiles and closure equations

$u(z)$ can be obtained from integration of the shear profile $\tau(z)$

- **Plug** zone: $\tau<\tau_y$
- **Shear** zone: $\tau>\tau\_z$

The **closure** relations need to relate the velocity profile with the depth-averaged variables ($\bar u$). Integrating $u(z)$ leads to:

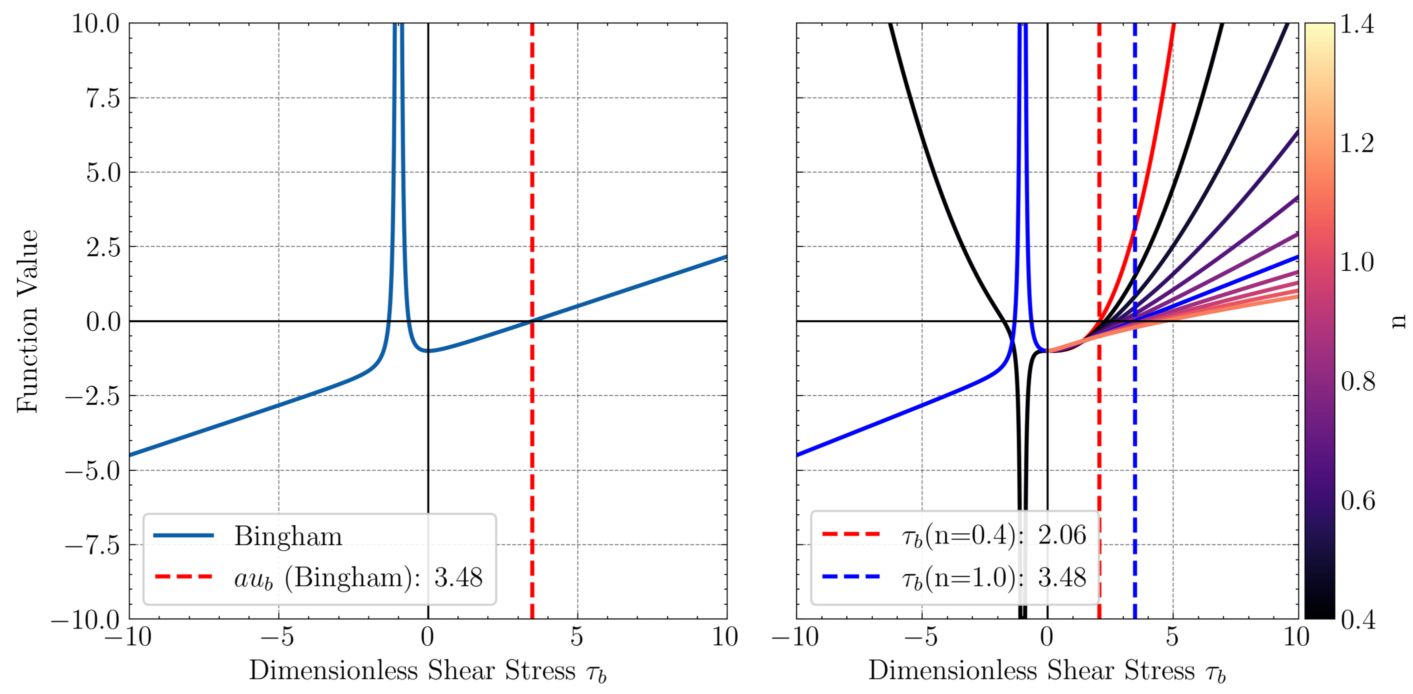


- **Bingham**:

$$2\tau_b^3+\tau_y^3-\tau_b^2\left(3\tau_y+6\mu_b\frac{\bar u}{h^2}\right)=0$$

- **Herschel-Bulkley:**

$$h\bar u=\frac{h^2n}{K^{1/n}\tau_b^2}\left[\frac{(\tau_b-\tau_y)^{\frac{2n+1}{n}}}{2n+1}+\frac{\tau_y}{n+1}(\tau_b-\tau_y)^{\frac{n+1}{n}}\right]$$



Modifications for non-hydrostatic pressure

Non-hydrostatic models enforce the Depth-Integrated Incompressibility Condition (**DICC**, $\vec\nabla\cdot\vec v=0$, [3]), improving the dispersive relation of the system.

$$\frac{\partial\tau}{\partial z}=-\rho g\frac{\partial h_s}{\partial x}-\frac{\partial p_{nh}}{\partial x}+\rho w\frac{\partial u}{\partial z}\Longrightarrow\frac{\partial\tau}{\partial z}=-\rho g\tan\theta+\rho w\frac{\tau}{\mu}\qquad\text{(Newtonian)}$$

Linear **ODE** of order 1, has analytical solution in terms of $\mathrm{erf}(z)=\int e^{-z^2}dz$

While **vertical velocities** can appear in a steady state, $\boldsymbol{p_{nh}}$ must be zero (vertical accelerations must vanish)

$$\frac{\partial\tau}{\partial z}=-\rho g\tan\theta+\rho w\frac{\tau-\tau_y}{\mu_b}\qquad\text{(Bingham)}$$

$$\frac{\partial\tau}{\partial z}=-\rho g\tan\theta+\rho w\left(\frac{\tau-\tau_y}{\mu_b}\right)^{1/n}\qquad\text{(Herschel-Bulkley)}$$

Calculate Variables (h, u, w) → Solve ODE $\tau(z)$ → Integrate $u(z)$ → Solve τ_b Calculate friction

Conclusions and future work

- Improved closure relations to account for **plug region**, and **non-linear** $\tau(z)$.
- Several models studied: Bingham, HB.
- **SWE** can be improved by **non-hydrostatic pressure** models.

- We Will study implementation strategies.
- Measure performance and comparison with simplifications.
- Study the performance of NHP models on steep slopes.

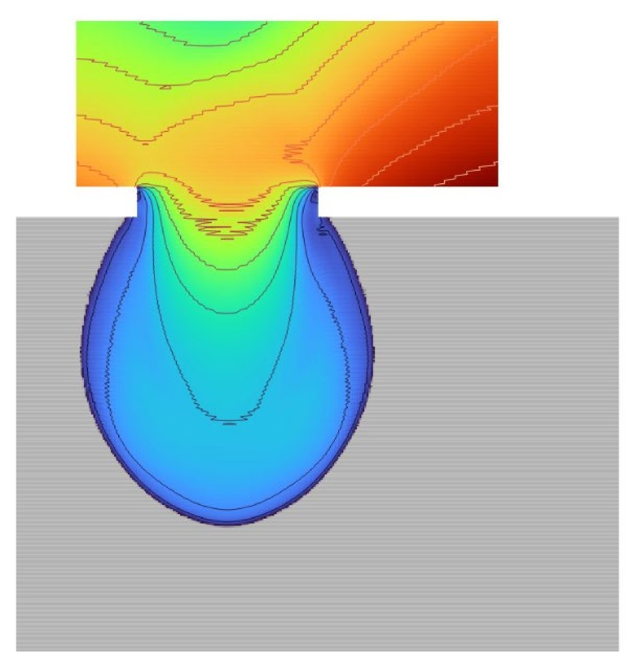


Figure: Mining pond dam-break simulation

Bibliography

[1] Garres-Díaz, J., Fernández-Nieto, E. D., Mangeney, A., & Morales de Luna, T. (2021). A weakly non-hydrostatic shallow model for dry granular flows. *Journal of Scientific Computing*, *86*(2), 25.

[2] Martínez-Aranda, S., Murillo, J., Morales-Hernández, M., & García-Navarro, P. (2022). Novel discretization strategies for the 2D non-Newtonian resistance term in geophysical shallow flows. *Engineering Geology*, *302*, 106625.

[3] Echeverribar, I., Brufau, P., & García-Navarro, P. (2023). Extension of a Roe-type Riemann solver scheme to model non-hydrostatic pressure shallow flows. *Applied Mathematics and Computation*, *440*, 127642.

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